CHAPTER 1

BASIC CONCEPTS

1. DIFFERENTIAL EQUATIONS►

A differential equation is an equation contains an unknown function and its derivative.

$$F\left(x, y,\ y',\ y'', \dots\dots\dots\dots, y^{(n)}\right) = 0$$

A differential equation is called an ordinary differential equation (O.D.E) if the unknown function depends on only one independent variable while if the unknown function depends on more than one variable, the differential equation is partial differential equation (P.D.E).

EXAMPLE 1►

(1) $\frac{dy}{dx} = 5x + 3$	O.D.E
(2) $e^x y'' + 2y' = 1$	O.D.E
(3) $4\frac{d^3y}{dx^3} + (sinx)\frac{d^2y}{dx^2} + 5xy = 0$	O.D.E
(4) $\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^5 + y^3\left(\frac{dy}{dx}\right)^2 = 5x$	O.D.E
(5) $\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2}$=0	P.D.E

■ ORDER OF D.E.►

The order of the differential equation is the highest derivative in the given D.E.

■ DEGREE OF D.E.►

The degree of the differential equation is the highest exponent of the highest derivative in the given D.E.

EXAMPLE 2►

D.E.	Order	Degree
(1)$(y'')^3 + y = 0$	2	3
(2) $4\frac{d^3y}{dx^3} + (sinx)\frac{d^2y}{dx^2} + 5xy = 0$	3	1
(3) $\left(\frac{d^4y}{dx^4}\right)^3 + 3y\left(\frac{dy}{dx}\right)^5 + y^3\left(\frac{dy}{dx}\right)^2 = 5x$	4	3
(4) $\left(\frac{\partial^3 y}{\partial x^3}\right)^2 + \frac{\partial y}{\partial t} = x^2 + t^2$	3	2
(5) $y'' = (y' - 4y)^{\frac{5}{3}}$	2	3

■ SOLUTION OF D.E.►

A solution of a D. E. is any function satisfying the D. E. (i.e.)

$y = y(x)$ is a solution for the D. E. $F\big(x, y,\ y',$

$y'', \dots\dots\dots\dots, y^{(n)}\big) = 0,$ if

(1) $y(x)$ is differentiable (n) times

(2) Satisfying the differential equation

$$F\big(x, y(x),\ y'(x),\ y''(x), \dots\dots\dots\dots, y^{(n)}(x)\big) = 0$$

EXAMPLE 3►

Prove that $y = e^{5x}$ is a solution for the equation

$$y' - 5y = 0$$

SOLUTION

Since $y = e^{5x} \rightarrow y' = 5e^{5x}$.

Then $y' - 5y = 5e^{5x} - 5e^{5x} = 0$.

EXAMPLE 4►

Prove that $y = c \sin x$ is a solution for the equation

$$y'' + y = 0$$

SOLUTION.

Since $y = c\ sin\ x \rightarrow y' = c\cos x \rightarrow y'' = -c\sin x$.

Then $y'' + y = -c\sin x + c\sin x = 0$.

EXAMPLE 5►

Prove that $ln\ y + \frac{x}{y} = c$, $y > 0$ is a solution for the equation $(y - x)y' + y = 0$

SOLUTION.

By differentiate the equation $ln\ y + \frac{x}{y} = c$ with respect to x

$$\frac{1}{y}y' + \frac{y - xy'}{y^2} = 0$$

$$yy' + y - xy' = 0, \quad then$$

$$(y - x)y' + y = 0$$

■ GENRAL SOLUTION AND PARTICULAR SOLUTION►

The general solution of a differential equation of order n is a function satisfies the differential equation and contains n constants.

The particular solution of a differential equation is a function satisfies the differential equation and do not contains any constants, and we get it by substitution specified values on the general solution.

EXAMPLE 6►

The general solution of the D.E.

$y''' - 5y' + 6y = 0$ is $y(x) = c_1 + c_2e^{2x} + c_3e^{3x}$,

and the particular solutions are

$y(x) = e^{2x} + e^{3x}$, $y(x) = 4 - 3e^{3x}$, $y(x) = 5 + 7e^{2x}$.

■ CONSTRUCTION OF D.E.►

In this part we obtain the differential equation by elimination the constants in the given solution

If we have the general solution of the differential equation of order (n) in the form

$$F(x, y, c_1, c_2, \dots\dots\dots, c_n) = 0 , \qquad (1)$$

where $c_1, c_2, \dots\dots\dots, c_n$ are constants.

(1) By differentiating the equation (1) n times we get n of equations contains $x, y, y', y'', \dots\dots. y^{(n)}$

(2) By elimination the constants $c_1, c_2, \dots\dots\dots, c_n$ from these equations we get the differential equation of order (n)

$g\left(x, y, y', y'', \dots\dots. y^{(n)}\right) = 0$

EXAMPLE 6 ►

Obtain the differential equation which has the solution

$y = c \sin x$ ... ... (1)

SOLUTION.

By differentiating the equation (1) we get

$y = c\cos x$ ... ... (2)

Dividing (2) over (1) yields the required differential equation

$y' = y\cot x$

EXAMPLE 7►

Find the differential equation which has the solution

$y = ae^{2x} + be^{3x}$ ... ... (1)

SOLUTION

By differentiating the equation (1) twice we get

$y' = 2ae^{2x} + 3be^{3x}$ ... ... (2)

$y'' = 4ae^{2x} + 9be^{3x}$ ... ... (3)

by solving the equation (2) and (3) together in a,b we have

$a = \frac{1}{2}e^{-2x}(3y' - y'')$ ... ... (4)

$b = \frac{1}{3}e^{-3x}(y'' - 2y)$ ... ... (5)

Substituting from (4) , (5) in (1) we get the differential equation

$y'' - 5y' + 6y = 0$

EXAMPLE 8 ►

Construct the differential equation which has the solution

$y = a + bx + x^2$

By differentiating the equation twice we get the differential equation $y'' = 2$

EXAMPLE 9►

Find the differential equation which has the solution

$y = ax + bx^2$ … … … … … … … … … … … . (1)

SOLUTION

By differentiating the equation (1) to eliminate the constants a,b as follows

$y' = a + 2bx$ … … … … … … … … … . (2)

$y'' = 2b$ … … … … … … … … … . … … … (3)

by solving the equation (2, 3) and Substituting in (1) we get the differential equation

$x^2y'' - 2xy' + 2y = 0$

■ INITIAL CONDITIONS AND BOUNDARY CONDITIONS►

In differential equation problems we give some conditions which must be satisfy the general solution and these conditions determine the appearing constants in the general solution and these conditions takes different forms such as

(1) if the conditions in the same point x_0 ,like

$$y(x_0) = a \quad , \quad y'(x_0) = b$$

Then these conditions called the initial conditions at x_0 and the differential equation and the initial conditions called **Initial value problem (I.V.P)**

(1) if the conditions in the different points ,like

$$y(x_1) = a \quad , \quad y(x_2) = b$$

Then these conditions called the boundary conditions and the differential equation and the boundary conditions called **boundary value problem (B.V.P)**

EXAMPLE 10►

Find the solution of the differential equation

$y' = 2x$, $y(2) = 3$

SOLUTION

By integral the differential equation

$$y(x) = x^2 + c$$

Using the condition in the equation we have

$$3 = 4 + c$$

Then the special solution is $y(x) = x^2 - 1$

EXAMPLE 11►

Find the solution of the initial value problem

$y'' = x$, $y(0) = 1$, $y'(0) = -1$

SOLUTION

By integral the differential equation twice we get

$$y'(x) = \frac{1}{2}x^2 + c_1$$

$$y(x) = \frac{1}{6}x^3 + c_1 x + c_2$$

Using the condition in the above equations we have

$$c_1 = -1 \ , \quad c_2 = 1$$

Then the special solution is

$y(x) = \frac{1}{6}x^3 - x + 1$

EXAMPLE 12►

Find the solution of the boundary value problem

$y'' = 6x + 2$, $y(0) = 2$, $y(2) = 8$

SOLUTION

By integral the differential equation

$$y(x) = x^3 + x^2 + c_1x + c_2$$

Using the condition in the equation we have

$$c_1 = -3 \ , \quad c_2 = 2$$

Then the special solution is

$$y(x) = x^3 + x^2 - 3x + 2$$

EXERCISES 1

1► Classify the following differential equations

(i) $(y'')^2 - 3yy' + x = 0$ (ii) $x^4y^{(4)} + xy''' = e^x$

(iii) $y'' = (3 - 2y')^{\frac{3}{2}}$ (iv) $(\frac{d^2y}{dx^2})^{5/3} + y = 2x$

2 ►Each of the following is a solution for the D.E. $y' - 3y = 6$

(i) $y = -2$ (ii) $y = 0$ (iii) $y = e^{3x} - 2$

(iv) $y = e^{2x} - 3$ (v) $y = 4e^{3x} - 2$

3► Each of the following is a solution for the D.E. $y' - 5y = 0$

(i) $y = 5$ (ii) $y = 5x$ (iii) $y = e^{5x}$

(iv) $y = x^5$ (v) $y = 2e^{5x}$ (vi) $y = 5e^{5x}$

4 ► Each of the following is a solution for the D.E. $y'' - y = 0$

(i) $y = e^x$ (ii) $y = \sin x$ (iii) $y = 4e^{-x}$

(iv) $y = 0$ (v) $y = \frac{1}{2}x^2 + 1$

5 ► Each of the following is a solution for the D.E. $y'' - 4y' + 4y = 0$

(i) $y = e^x$ (ii) $y = e^{2x}$ (iii) $y = e^{2x} + e^x$

(iv) $y = xe^{2x} + e^x$ (v) $y = e^{2x} - xe^x$

6 ►Obtain the differential equation by elimination constants for the following equations

(1) $y = ax^2 + be^x$ (2) $y = ax + b$

(3) $y = a\cos x + bx$ (4) $y = ae^{2x} + be^x$

(5) $y = x^3 + ax + b$ (6) $y = ae^x + bxe^x$

(7) $y = a\sin x + bx$ (8) $y = ae^{-x} + be^{3x}$

(9) $y = ae^x$ (10) $y = a\sin x$

(11) $y = a\sin(4x) + b\cos(4x)$ (12) $\ln y = ax^2 + b$